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Well-posedness of constrained minimization problems via saddle-points

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Abstract In this paper, it is proved a very general well-posedness result for a class of constrained minimization problems of which the following is a particular case: Let X be a Hausdorff topological space and let $J, \Phi: X \to \mathbb{R}$ be two non-constant functions such that, for each $\lambda \in \mathbb{R}$, the function $J + \lambda\Phi$ has sequentially compact sub-level sets and admits a unique global minimum in X. Then, for each $r \in]\inf_X \Phi, \sup_X \Phi[$, the restriction of J to $\Phi^{-1}(r)$ has a unique global minimum, say $\hat{x}_r$, toward which every minimizing sequence converges. Moreover, the functions $r \to \hat{x}_r$ and $r \to J(\hat{x}_r)$ are continuous in $]\inf_X \Phi, \sup_X \Phi[$.

Keywords Constrained minimization problem · Well-posedness · Minimax · Saddle-point

Here and in the sequel, X is a Hausdorff topological space, J, Φ are two real-valued functions defined in X, and a, b are two numbers in $[-\infty, +\infty]$, with $a < b$.

If $a \in \mathbb{R}$ (resp. $b \in \mathbb{R}$), we denote by M_a (resp. M_b) the set of all global minima of the function $J + a\Phi$ (resp. $J + b\Phi$), while if $a = -\infty$ (resp. $b = +\infty$), M_a (resp. M_b) stands for the empty set. We adopt the conventions $\inf \emptyset = +\infty$, $\sup \emptyset = -\infty$. We also set

$$\alpha := \max \left\{ \inf_X \Phi, \sup_{M_b} \Phi \right\},$$

$$\beta := \min \left\{ \sup_X \Phi, \inf_{M_a} \Phi \right\}.$$

Note that, by Proposition 1 below, one has $\alpha \leq \beta$.

As usual, given a function $f: X \to \mathbb{R}$ and a set $C \subseteq X$, we say that the problem of minimizing f over C is well-posed if the following two conditions hold:

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- the restriction of f to C has a unique global minimum, say $\hat{x}$;
- every sequence $\{x_n\}$ in C such that $\lim_{n\to\infty} f(x_n) = \inf_C f$, converges to $\hat{x}$.

A set of the type $\{x \in X\colon f(x) \leq r\}$ is said to be a sub-level set of f. Clearly, when the sub-level sets of f are sequentially compact, the problem of minimizing f over a sequentially closed set C is well-posed if and only if $f_{|C}$ has a unique global minimum.

The aim of the present paper is to establish the following result:

Theorem 1 *Assume that $\alpha < \beta$ and that, for each $\lambda \in]a, b[$, the function $J + \lambda\Phi$ has sequentially compact sub-level sets and admits a unique global minimum in X.*

Then, for each $r \in]\alpha, \beta[$, the problem of minimizing J over $\Phi^{-1}(r)$ is well-posed.

Moreover, if we denote by $\hat{x}_r$ the unique global minimum of $J_{|\Phi^{-1}(r)}$ $(r \in]\alpha, \beta[)$, the functions $r \to \hat{x}_r$ and $r \to J(\hat{x}_r)$ are continuous in $]\alpha, \beta[$.

Theorem 1 should be regarded as the definitive abstract result coming out from the saddle-point method developed in Refs. [4–7], in specific settings.

The main tool used to prove Theorem 1 is provided by the following mini–max result:

Theorem 2 *Let $I \subseteq \mathbb{R}$ be an interval and f a real-valued function defined in $X \times I$. Assume that there exist a number $\rho^* > \sup_I \inf_X f$ and a point $\hat{\lambda} \in I$ such that, for each $\rho \leq \rho^*$, the following conditions hold:*

(i) the set $\{\lambda \in I\colon f(x, \lambda) > \rho\}$ is connected for all $x \in X$;
(ii) the set $\{x \in X\colon f(x, \lambda) \leq \rho\}$ is sequentially closed for all $\lambda \in I$ and sequentially compact for $\lambda = \hat{\lambda}$;
(iii) for each compact interval $T \subseteq I$ for which $\sup_T \inf_X f < \rho$, there exists a continuous function $\varphi\colon T \to X$ such that $f(\varphi(\lambda), \lambda) < \rho$ for all $\lambda \in T$.
Then, one has

$$\sup_{\lambda\in I} \inf_{x\in X} f(x, \lambda) = \inf_{x\in X} \sup_{\lambda\in I} f(x, \lambda).$$

Proof We strictly follow the proof of Theorem 2 of [3]. First, fix a non-decreasing sequence $\{I_n\}$ of compact sub-intervals of I, with $\hat{\lambda} \in I_1$, such that $\cup_{n\in\mathbf{N}} I_n = I$. Now, fix $n \in \mathbf{N}$. We claim that

$$\sup_{\lambda\in I_n} \inf_{x\in X} f(x, \lambda) = \inf_{x\in X} \sup_{\lambda\in I_n} f(x, \lambda). \tag{1}$$

Arguing by contradiction, suppose that

$$\sup_{\lambda\in I_n} \inf_{x\in X} f(x, \lambda) < \inf_{x\in X} \sup_{\lambda\in I_n} f(x, \lambda).$$

Fix ρ satisfying

$$\sup_{\lambda\in I_n} \inf_{x\in X} f(x, \lambda) < \rho < \min\left\{\rho^*, \inf_{x\in X} \sup_{\lambda\in I_n} f(x, \lambda)\right\}.$$

Set

$$S = \{(x, \lambda) \in X \times I_n\colon f(x, \lambda) < \rho\}$$

as well as, for each $\lambda \in I_n$,

$$S^\lambda = \{x \in X\colon (x, \lambda) \in S\}.$$

Since $\sup_{I_n} \inf_X f < \rho$, one has $S^\lambda \neq \emptyset$ for all $\lambda \in I_n$. Let $I_n = [a_n, b_n]$. Put

$$A = \left\{ (x, \lambda) \in S\colon \lambda < b_n\ ,\ \sup_{s \in]\lambda, b_n]} f(x, s) > \rho \right\}$$

and

$$B = \left\{ (x, \lambda) \in S\colon \lambda > a_n\ ,\ \sup_{s \in [a_n, \lambda[} f(x, s) > \rho \right\}.$$

Observe that $S^{a_n} \times \{a_n\} \subseteq A$ and $S^{b_n} \times \{b_n\} \subseteq B$. Indeed, let $x_1 \in S^{a_n}$ and $x_2 \in S^{b_n}$. Since $\rho < \inf_X \sup_{I_n} f$, there are $t, s \in I_n$ such that $\min\{f(x_1, t), f(x_2, s)\} > \rho$. Since $\sup\{f(x_1, a_n), f(x_2, b_n)\} < \rho$, it follows that $t > a_n$ and $s < b_n$. Consequently, $(x_1, a_n) \in A$ and $(x_2, b_n) \in B$. Furthermore, observe that if $(x_0, \lambda_0) \in A$ and if $\mu \in]\lambda_0, b_n]$ is such that $f(x_0, \mu) > \rho$, then, in view of (ii), the set

$$(\{x \in X\colon f(x, \mu) > \rho\} \times [a_n, \mu[) \cap S$$

is sequentially open in S, contains (x_0, λ_0) and is contained in A. In other words, A is sequentially open in S. Analogously, it is seen that B is sequentially open in S. We now prove that $S = A \cup B$. Indeed, let $(x, \lambda) \in S \setminus A$. We have seen above that $S^{a_n} \times \{a_n\} \subseteq A$, and so $\lambda > a_n$. If $\lambda = b_n$, the fact that $(x, \lambda) \in B$ has been likewise proved above. Suppose $\lambda < b_n$. Thus, we have $\sup_{s \in]\lambda, b_n]} f(x, s) \leq \rho$. From this, it clearly follows that $\sup_{s \in [a_n, \lambda[} f(x, s) > \rho$ (note that $f(x, \lambda) < \rho$), and so $(x, \lambda) \in B$. Furthermore, we have $A \cap B = \emptyset$. Indeed, if $(x_1, \lambda_1) \in A \cap B$, there would be $t_1, s_1 \in I_n$, with $t_1 < \lambda_1 < s_1$, such that $\min\{f(x_1, t_1), f(x_1, s_1)\} > \rho$. By (i), the set $\{s \in I\colon f(x_1, s) > \rho\}$ is an interval, and so we would have $f(x_1, \lambda_1) > \rho$, against the fact that $(x_1, \lambda_1) \in S$. Now, in view of (iii), consider a continuous function $\varphi\colon I_n \to X$ such that

$$f(\varphi(\lambda), \lambda) < \rho$$

for all $\lambda \in I_n$. Let $h\colon I_n \to X \times I_n$ be defined by setting

$$h(\lambda) = (\varphi(\lambda), \lambda)$$

for all $\lambda \in I_n$. Since h is continuous, the set $h(I_n)$ is sequentially connected ([1], Theorem 2.2). But, having in mind that $h(I_n) \subseteq S$ and that $h(I_n)$ meets both A and B (since $h(a_n) \in A$ and $h(b_n) \in B$), the properties of A, B proved above would imply that $h(I_n)$ is sequentially disconnected, a contradiction. So, (1) holds. Finally, let us prove the theorem. Again arguing by contradiction, suppose that

$$\sup_{\lambda \in I} \inf_{x \in X} f(x, \lambda) < \inf_{x \in X} \sup_{\lambda \in I} f(x, \lambda).$$

Choose r satisfying

$$\sup_{\lambda \in I} \inf_{x \in X} f(x, \lambda) < r < \min\left\{ \rho^*, \inf_{x \in X} \sup_{\lambda \in I} f(x, \lambda) \right\}.$$

For each $n \in \mathbf{N}$, put

$$C_n = \left\{ x \in X\colon \sup_{\lambda \in I_n} f(x, \lambda) \leq r \right\}.$$

Note that $C_n \neq \emptyset$. Indeed, otherwise, we would have

$$r \leq \inf_{x \in X} \sup_{\lambda \in I_n} f(x, \lambda) = \sup_{\lambda \in I_n} \inf_{x \in X} f(x, \lambda) \leq \sup_{\lambda \in I} \inf_{x \in X} f(x, \lambda).$$

Consequently, $\{C_n\}$ is a non-increasing sequence of non-empty sequentially closed subsets of the sequentially compact set $\{x \in X\colon f(x, \hat{\lambda}) \leq \rho^*\}$. Therefore, one has $\cap_{n \in \mathbf{N}} C_n \neq \emptyset$. Let $x^* \in \cap_{n \in \mathbf{N}} C_n$. Then, one has

$$\sup_{\lambda \in I} f(x^*, \lambda) = \sup_{n \in \mathbf{N}} \sup_{\lambda \in I_n} f(x^*, \lambda) \leq r$$

and so

$$\inf_{x \in X} \sup_{\lambda \in I} f(x, \lambda) \leq r,$$

a contradiction. The proof is complete. □

We will also use the following proposition.

Proposition 1 ([4], Proposition 1) *Let Y be a non-empty set, $f, g\colon Y \to \mathbb{R}$ two functions, and λ, μ two real numbers, with $\lambda < \mu$. Let $\hat{y}_\lambda$ be a global minimum of the function $f + \lambda g$ and let $\hat{y}_\mu$ be a global minimum of the function $f + \mu g$.*

Then, one has

$$g(\hat{y}_\mu) \leq g(\hat{y}_\lambda).$$

If either $\hat{y}_\lambda$ or $\hat{y}_\mu$ is strict and $\hat{y}_\lambda \neq \hat{y}_\mu$, then

$$g(\hat{y}_\mu) < g(\hat{y}_\lambda).$$

Proof of Theorem 1 First, for each $\lambda \in]a, b[$, denote by $\hat{y}_\lambda$ the unique global minimum in X of $J + \lambda\Phi$. Let us prove that the function $\lambda \to \hat{y}_\lambda$ is continuous in $]a, b[$. To this end, fix $\lambda^* \in]a, b[$. Let $\{\lambda_n\}$ be any sequence in $]a, b[$ converging to λ^* and let $[c, d] \subset]a, b[$ be a compact interval containing $\{\lambda_n\}$. Fix $\rho > \sup_{n \in \mathbf{N}} \inf_{x \in X}(J(x) + \lambda_n \Phi(x))$. Clearly, we have

$$\bigcup_{\lambda \in [c,d]} \{x \in X\colon J(x) + \lambda\Phi(x) \leq \rho\}$$

$$\subseteq \{x \in X\colon J(x) + c\Phi(x) \leq \rho\} \cup \{x \in X\colon J(x) + d\Phi(x) \leq \rho\}.$$

From this, due to the choice of ρ, we infer that the sequence $\{\hat{y}_{\lambda_n}\}$ is contained in the set on the right-hand side which is clearly sequentially compact. Hence, there is a subsequence $\{\hat{y}_{\lambda_{n_k}}\}$ converging to some $y^* \in X$. Taking into account that the sequence $\{\Phi(\hat{y}_{\lambda_{n_k}})\}$ is bounded (by Proposition 1) and that the function $J + \lambda^*\Phi$ is sequentially lower semicontinuous, for each $x \in X$, we then have

$$\begin{aligned} J(y^*) + \lambda^*\Phi(y^*) &\leq \liminf_{k \to \infty} (J(\hat{y}_{\lambda_{n_k}}) + \lambda^*\Phi(\hat{y}_{\lambda_{n_k}})) \\ &= \liminf_{k \to \infty} (J(\hat{y}_{\lambda_{n_k}}) + \lambda_{n_k}\Phi(\hat{y}_{\lambda_{n_k}}) + (\lambda^* - \lambda_{n_k})\Phi(\hat{y}_{\lambda_{n_k}})) \\ &= \liminf_{k \to \infty} (J(\hat{y}_{\lambda_{n_k}}) + \lambda_{n_k}\Phi(\hat{y}_{\lambda_{n_k}})) \leq \lim_{k \to \infty} (J(x) + \lambda_{n_k}\Phi(x)) \\ &= J(x) + \lambda^*\Phi(x) \end{aligned}$$

Hence y^* is the global minimum of $J + \lambda^*\Phi$, that is $y^* = \hat{y}_{\lambda^*}$, which shows the continuity of $\lambda \to \hat{y}_\lambda$ at λ^*. Now, fix $r \in]\alpha, \beta[$ and consider the function $f : X \times \mathbb{R} \to \mathbb{R}$ defined by

$$f(x, \lambda) = J(x) + \lambda(\Phi(x) - r)$$

for all $(x, \lambda) \in X \times \mathbb{R}$. Clearly, the restriction of the function f to $X \times]a, b[$ satisfies all the assumptions of Theorem 2. In particular, (iii) is satisfied taking $\varphi(\lambda) = \hat{y}_\lambda$. Consequently, we have

$$\sup_{\lambda \in]a,b[} \inf_{x \in X} (J(x) + \lambda(\Phi(x) - r)) = \inf_{x \in X} \sup_{\lambda \in]a,b[} (J(x) + \lambda(\Phi(x) - r)). \tag{2}$$

Note that

$$\begin{aligned} \sup_{\lambda \in]a,b[} \inf_{x \in X} f(x, \lambda) &\le \sup_{\lambda \in [a,b] \cap \mathbb{R}} \inf_{x \in X} f(x, \lambda) \\ &\le \inf_{x \in X} \sup_{\lambda \in [a,b] \cap \mathbb{R}} f(x, \lambda) = \inf_{x \in X} \sup_{\lambda \in]a,b[} f(x, \lambda) \end{aligned}$$

and so from (2) it follows

$$\sup_{\lambda \in [a,b] \cap \mathbb{R}} \inf_{x \in X} (J(x) + \lambda(\Phi(x) - r)) = \inf_{x \in X} \sup_{\lambda \in [a,b] \cap \mathbb{R}} (J(x) + \lambda(\Phi(x) - r)). \tag{3}$$

Now, observe that the function $\inf_{x \in X} f(x, \cdot)$ is upper semicontinuous in $[a, b] \cap \mathbb{R}$ and that

$$\lim_{\lambda \to +\infty} \inf_{x \in X} f(x, \lambda) = -\infty$$

if $b = +\infty$ (since $r > \inf_X \Phi$), and

$$\lim_{\lambda \to -\infty} \inf_{x \in X} f(x, \lambda) = -\infty$$

if $a = -\infty$ (since $r < \sup_X \Phi$). From this, it clearly follows that there exists $\hat{\lambda}_r \in [a, b] \cap \mathbb{R}$ such that

$$\inf_{x \in X} f(x, \hat{\lambda}_r) = \sup_{\lambda \in [a,b] \cap \mathbb{R}} \inf_{x \in X} f(x, \lambda).$$

Since

$$\sup_{\lambda \in [a,b] \cap \mathbb{R}} f(x, \lambda) = \sup_{\lambda \in]a,b[} f(x, \lambda)$$

for all $x \in X$, the sub-level sets of the function $\sup_{\lambda \in [a,b] \cap \mathbb{R}} f(\cdot, \lambda)$ are sequentially compact. Hence, there exists $\hat{x}_r \in X$ such that

$$\sup_{\lambda \in [a,b] \cap \mathbb{R}} f(\hat{x}_r, \lambda) = \inf_{x \in X} \sup_{\lambda \in [a,b] \cap \mathbb{R}} f(x, \lambda).$$

Then, thanks to (3), $(\hat{x}_r, \hat{\lambda}_r)$ is a saddle-point of f, that is

$$\begin{aligned} J(\hat{x}_r) + \hat{\lambda}_r(\Phi(\hat{x}_r) - r) = \inf_{x \in X} (J(x) + \hat{\lambda}_r(\Phi(x) - r)) &= J(\hat{x}_r) \\ + \sup_{\lambda \in [a,b] \cap \mathbb{R}} \lambda(\Phi(\hat{x}_r) - r). \end{aligned} \tag{4}$$

First of all, from (4) it follows that $\hat{x}_r$ is a global minimum of $J + \hat{\lambda}_r\Phi$. We now show that $\Phi(\hat{x}_r) = r$. We distinguish four cases.

- $a = -\infty$ and $b = +\infty$. In this case, the equality $\Phi(\hat{x}_r) = r$ follows from the fact that $\sup_{\lambda \in \mathbb{R}} \lambda(\Phi(\hat{x}_r) - r)$ is finite.
- $a > -\infty$ and $b = +\infty$. In this case, the finiteness of $\sup_{\lambda \in [a, +\infty[} \lambda(\Phi(\hat{x}_r) - r)$ implies that $\Phi(\hat{x}_r) \leq r$. But, if $\Phi(\hat{x}_r) < r$, from (4), we would infer that $\hat{\lambda}_r = a$ and so $\hat{x}_r \in M_a$. This would imply $\inf_{M_a} \Phi < r$, contrary to the choice of r.
- $a = -\infty$ and $b < +\infty$. In this case, the finiteness of $\sup_{\lambda \in]-\infty, b]} \lambda(\Phi(\hat{x}_r) - r)$ implies that $\Phi(\hat{x}_r) \geq r$. But, if $\Phi(\hat{x}_r) > r$, from (4) again, we would infer $\hat{\lambda}_r = b$, and so $\hat{x}_r \in M_b$. Therefore, $\sup_{M_b} \Phi > r$, contrary to the choice of r.
- $-\infty < a$ and $b < +\infty$. In this case, if $\Phi(\hat{x}_r) \neq r$, as we have just seen, we would have either $\inf_{M_a} \Phi < r$ or $\sup_{M_b} \Phi > r$, contrary to the choice of r.

Having proved that $\Phi(\hat{x}_r) = r$, we also get that $\hat{\lambda}_r \in]a, b[$. Indeed, if $\hat{\lambda}_r \in \{a, b\}$, we would have either $\hat{x}_r \in M_a$ or $\hat{x}_r \in M_b$ and so either $\inf_{M_a} \Phi \leq r$ or $\sup_{M_b} \Phi \geq r$, contrary to the choice of r. From (4) once again, we furthermore infer that any global minimum of $J_{|\Phi^{-1}(r)}$ (and $\hat{x}_r$ is so) is a global minimum of $J + \hat{\lambda}_r \Phi$ in X. But, since $\hat{\lambda}_r \in]a, b[$, $J + \hat{\lambda}_r \Phi$ has exactly one global minimum in X which, therefore, coincides with $\hat{x}_r$. Since the sub-level sets of $J + \hat{\lambda}_r \Phi$ are sequentially compact, we then conclude that any minimizing sequence in X for $J + \hat{\lambda}_r \Phi$ converges to $\hat{x}_r$. But any minimizing sequence in $\Phi^{-1}(r)$ for J is a minimizing sequence for $J + \hat{\lambda}_r \Phi$, and so it converges to $\hat{x}_r$. Consequently, the problem of minimizing J over $\Phi^{-1}(r)$ is well-posed, as claimed.

Now, let us prove the other assertions made in thesis. By Proposition 1, it clearly follows that the function $\lambda \to \Phi(\hat{y}_\lambda)$ is non-increasing in $]a, b[$ and that its range is contained in $[\alpha, \beta]$. On the other hand, by the first assertion of the thesis, this range contains $]\alpha, \beta[$. Of course, from this it follows that the function $\lambda \to \Phi(\hat{y}_\lambda)$ is continuous in $]a, b[$. Now, observe that the function $\lambda \to \inf_{x \in X}(J(x) + \lambda\Phi(x))$ is concave and hence continuous in $]a, b[$. This, in particular, implies that the function $\lambda \to J(\hat{y}_\lambda)$ is continuous in $]a, b[$. Now, for each $r \in]\alpha, \beta[$, put

$$\Lambda_r = \{\lambda \in]a, b[: \Phi(\hat{y}_\lambda) = r\}.$$

Let us prove that the multifunction $r \to \Lambda_r$ is upper semicontinuous in $]\alpha, \beta[$. Of course, it is enough to show that the restriction of the multifunction to any bounded open sub-interval of $]\alpha, \beta[$ is upper semicontinuous. So, let $s, t \in]\alpha, \beta[$, with $s < t$. Let $\mu, \nu \in]a, b[$ be such that $\Phi(\hat{y}_\mu) = t$, $\Phi(\hat{y}_\nu) = s$. By Proposition 1, we have

$$\bigcup_{r \in]s, t[} \Lambda_r \subseteq [\mu, \nu].$$

Then, to show that the restriction of multifunction $r \to \Lambda_r$ to $]s, t[$ is upper semicontinuous, it is enough to prove that its graph is closed in $]s, t[\times [\mu, \nu]$ ([2], Theorem 7.1.16). But, this latter fact follows immediately from the continuity of the function $\lambda \to \Phi(\hat{y}_\lambda)$. At this point, we observe that, for each $r \in]\alpha, \beta[$, the function $\lambda \to \hat{y}_\lambda$ is constant in Λ_r. Indeed, let $\lambda, \mu \in \Lambda_r$ with $\lambda \neq \mu$. If it was $\hat{y}_\lambda \neq \hat{y}_\mu$, by Proposition 1 it would follow

$$r = \Phi(\hat{y}_\lambda) \neq \Phi(\hat{y}_\mu) = r$$

an absurd. Hence, the function $r \to \hat{x}_r$, as composition of the upper semicontinuous multifunction $r \to \Lambda_r$ and the continuous function $\lambda \to \hat{y}_\lambda$, is continuous. Analogously, the continuity of the function $r \to J(\hat{x}_r)$ follows observing that it is the composition of $r \to \Lambda_r$ and the continuous function $\lambda \to J(\hat{y}_\lambda)$. The proof is complete. □

Remark 1 We want to point out that, under the assumptions of Theorem 1, we have actually proved that, for each $r \in]\alpha, \beta[$, there exists $\hat{\lambda}_r \in]a, b[$ such that the unique global minimum of $J + \hat{\lambda}_r \Phi$ belongs to $\Phi^{-1}(r)$.

When $a \geq 0$, we can obtain a conclusion dual to that of Theorem 1, under the same key assumption.

Theorem 3 *Let $a \geq 0$. Assume that, for each $\lambda \in]a, b[$, the function $J + \lambda\Phi$ has sequentially compact sub-level sets and admits a unique global minimum in X.*

Set

$$\gamma := \max\left\{\inf_X J, \sup_{\hat{M}_a} J\right\},$$

$$\delta := \min\left\{\sup_X J, \inf_{\hat{M}_b} J\right\},$$

where

$$\hat{M}_a = \begin{cases} M_a & \text{if } a > 0, \\ \emptyset & \text{if } a = 0, \end{cases}$$

$$\hat{M}_b = \begin{cases} M_b & \text{if } b < +\infty, \\ \Phi^{-1}(\inf_X \Phi) & \text{if } b = +\infty. \end{cases}$$

Assume that $\gamma < \delta$.

Then, for each $r \in]\gamma, \delta[$, the problem of minimizing Φ over $J^{-1}(r)$ is well-posed.

Moreover, if we denote by $\tilde{x}_r$ the unique global minimum of $\Phi_{|J^{-1}(r)}$ $(r \in]\gamma, \delta[)$, the functions $r \to \tilde{x}_r$ and $r \to \Phi(\tilde{x}_r)$ are continuous in $]\gamma, \delta[$.

Proof Let $\mu \in]b^{-1}, a^{-1}[$. Then, since $\mu^{-1} \in]a, b[$ and

$$\Phi + \mu J = \mu(J + \mu^{-1}\Phi)$$

we clearly have that the function $\Phi + \mu J$ has sequentially compact sub-level sets and admits a unique global minimum. At this point, the conclusion follows applying Theorem 1 with the roles of J an Φ interchanged. □

We now state the version of Theorem 1 obtained in the setting of a reflexive Banach space endowed with the weak topology.

Theorem 4 *Let X be a sequentially weakly closed set in a reflexive real Banach space. Assume that $\alpha < \beta$ and that, for each $\lambda \in]a, b[$, the function $J + \lambda\Phi$ is sequentially weakly lower semicontinuous, has bounded sub-level sets and has a unique global minimum in X.*

Then, for each $r \in]\alpha, \beta[$, the problem of minimizing J over $\Phi^{-1}(r)$ is well-posed in the weak topology.

Moreover, if we denote by $\hat{x}_r$ the unique global minimum of $J_{|\Phi^{-1}(r)}$ $(r \in]\alpha, \beta[)$, the functions $r \to \hat{x}_r$ and $r \to J(\hat{x}_r)$ are continuous in $]\alpha, \beta[$, the first one in the weak topology.

Proof Our assumptions clearly imply that, for each $\lambda \in]a, b[$, the sub-level sets of $J + \lambda\Phi$ are sequentially weakly compact, by the Eberlein–Smulyan theorem. Hence, considering X with the relative weak topology, we are allowed to apply Theorem 1, from which the conclusion directly follows. □

Analogously, from Theorem 3 we get

Theorem 5 *Let $a \geq 0$ and let X be a sequentially weakly closed set in a reflexive real Banach space. Assume that, for each $\lambda \in]a, b[$, the function $J + \lambda\Phi$ is sequentially weakly lower semicontinuous, has bounded sub-level sets and has a unique global minimum in X. Assume also that $\gamma < \delta$, where γ, δ are defined as in Theorem 3.*

Then, for each $r \in]\gamma, \delta[$, the problem of minimizing Φ over $J^{-1}(r)$ is well-posed in the weak topology.

Moreover, if we denote by $\tilde{x}_r$ the unique global minimum of $\Phi_{|J^{-1}(r)}$ $(r \in]\gamma, \delta[)$, the functions $r \to \tilde{x}_r$ and $r \to \Phi(\tilde{x}_r)$ are continuous in $]\gamma, \delta[$, the first one in the weak topology.

Finally, it is worth noticing that Theorem 1 also offers the perspective of a novel way of seeing whether a given function possesses a global minimum. Let us formalize this using Remark 1.

Theorem 6 *Assume that $b > 0$ and that, for each $\lambda \in]0, b[$, the function $J + \lambda\Phi$ has sequentially compact sub-level sets and admits a unique global minimum, say $\hat{y}_\lambda$. Assume also that*

$$\lim_{\lambda \to 0^+} \Phi(\hat{y}_\lambda) < \sup_X \Phi. \tag{5}$$

Then, one has

$$\lim_{\lambda \to 0^+} \Phi(\hat{y}_\lambda) = \inf_M \Phi,$$

where M is the set of all global minima of J in X.

Proof We already know that the function $\lambda \to \Phi(\hat{y}_\lambda)$ is non-increasing in $]a, b[$ and that its range is contained in $[\alpha, \beta]$. We claim that

$$\beta = \lim_{\lambda \to 0^+} \Phi(\hat{y}_\lambda).$$

Assume the contrary. Let us apply Theorem 1, with $a = 0$ (so, $M_0 = M$), using the conclusion pointed out in Remark 1. Choose r satisfying

$$\lim_{\lambda \to 0^+} \Phi(\hat{y}_\lambda) < r < \beta.$$

Then, (since also $\alpha < r$) it would exist $\hat{\lambda}_r \in]0, b[$ such that $\Phi(\hat{y}_{\hat{\lambda}_r}) = r$, contrary to the choice of r. At this point, the conclusion follows directly from (5). □

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